An exact interpolating index for 4d $\mathcal{N} = 2$ SCFTs

[25xx.xxxxx]

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under the supervision of I. Yaakov

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- Susy on curved spaces
- The Donaldson-Witten twist of 4d $\mathcal{N} = 2$
- The Superconformal Index of 4d $\mathcal{N} = 2$
- The Interpolating Index
- Summary, open questions, and future directions

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Minimal coupling

Consider a supersymmetric theory defined on $\mathbb{R}^4$. We now want to place that theory on an oriented Riemannian manifold X.

Does minimal coupling preserve supersymmetry on X?

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Does minimal coupling preserve supersymmetry on X?

Ingredients:

- X must be Spin (admit a Spin-bundle $\mathcal{S}X$)
- X must admit non-trivial solutions to the Killing spinor equation $\nabla\xi = 0$, $\xi \in \Gamma(\mathcal{S}X)$

Answer: Yes but for very few X.

eg. 4d compact: only T^4 and $K3$-surfaces with Ricci-flat Kähler metrics.

Twisting

Take a supersymmetric theory on an oriented Riemannian manifold X, with R-symmetry bundle P_R.

Identify a sub-bundle of the spin bundle $\mathcal{S}X$ with the R-symmetry bundle P_R, together with their connections.

Eg. For 4d $\mathcal{N} = 4$ theories, the homomorphism

$$\mathrm{Spin}(4) \cong SU(2)_+ \times SU(2)_- \longrightarrow SU(4)_R \tag{1}$$

determines the twist (Vafa-Witten, Kapustin-Witten or Marcus).

Eg. 4d $\mathcal{N} = 2$ SCFT for which the R-symmetry group is $SU(2)_R \times U(1)_R$ and the spin bundle separates into two $SU(2)$ bundles, $\mathcal{S}X^{\pm}$. The spinors now have an $SU(2)_R$ index and the Killing spinor equation generalises to

$$d\xi^i + \frac{1}{4}\omega^{ab}\gamma_{ab} \cdot \xi^i + \frac{1}{2}A^{SU(2)_R\,i}{}_j\xi^i = 0. \tag{2}$$

Let's identify $\mathrm{diag}(SU(2)_+ \times SU(2)_R)$ with $SU(2)_+$ via

$$A^{SU(2)} = \frac{1}{2}\omega^{ab}\sigma_{ab} \tag{3}$$

This always preserves at least one supercharge, let's call it $\mathcal{Q}$.

Twisting

1 Susy on curved spaces

One of the original supercharges $\mathcal{Q}$ becomes a singlet of the new Lorentz group and $T_{\mu\nu}$ becomes $\mathcal{Q}$-exact,

$$d_{\text{Met}} \int \mathcal{D}\Phi e^{-S_{\text{twisted}}[\Phi; g_{\mu\nu}]} = 0. \tag{4}$$

The content of theory decomposes into a cohomology of $\mathcal{Q}$. We get a *cohomological TQFT*.

Ingredients: X can be any smooth manifold and the partition function

$$Z_{\text{DW}}[g_{\mu\nu}] = \int \mathcal{D}\Phi e^{-S_{\text{twisted}}[\Phi; g_{\mu\nu}]} \tag{5}$$

localizes to the volume of the moduli space of instantons.
$\hookrightarrow$ divergence can be regularized (eg. Ω-deformation/T^2-equivariant volume)

The systematic way

A la Festuccia-Seiberg [Festuccia, Seiberg 2011]

1. Couple theory to off-shell supergravity
2. Take the rigid limit ($M_P \to \infty$)

The Lagrangian density then looks like

$$\mathcal{L}_X(\Phi, \mathcal{H}) = \mathcal{L}_{\mathbb{R}^d}(\Phi) + (\mathcal{H}, \mathcal{S}(\Phi)) + \mathcal{O}(\Phi) \tag{6}$$

$\mathcal{H}$: sugra multiplet, $\mathcal{S}$: susy multiplet containing $T_{\mu\nu}$

Supersymmetry is conserved iff $\delta_\xi \mathcal{H} = 0$.

Example: 4d $\mathcal{N} = 1$ [Closset, Dumitrescu, Festuccia, Komargodski 2014]

1. $\mathcal{R}$-multiplet $\to$ new minimal supergravity

$$\begin{aligned} \mathcal{R}_\mu &= (j_\mu^{(R)}, S_{\alpha\mu}, \tilde{S}^{\dot{\alpha}}{}_\mu, T_{\mu\nu}, \mathcal{F}_{\mu\nu}) \\ \mathcal{H}_\mu &= (A_\mu^{(R)}, \Psi_{\alpha\mu}, \tilde{\Psi}^{\dot{\alpha}}{}_\mu, h_{\mu\nu}, \mathcal{B}_{\mu\nu}) \\ \Delta\mathcal{L} &= 2\int d^4\theta \mathcal{R}_\mu \mathcal{H}^\mu \end{aligned} \tag{7}$$

2. Ferrara-Zumino multiplet $\to$ old minimal supergravity

When the theory is conformal, it is natural to couple to *conformal supergravity*.
With a 4d $\mathcal{N}=2$ SCFT, we wish to couple to the Weyl multiplet of $\mathcal{N}$-extended conformal sugra

[de Wit, Reys 2017]

$$e_\mu{}^a \quad \psi_\mu{}^i \quad b_\mu \quad A_\mu^{U(1)_R} \quad A_\mu^{SU(2)_R} \quad T_{ab} \quad \chi^i \quad D \tag{8}$$

$$\Delta\mathcal{L} = J^\mu_e{}_a\, e_\mu{}^a + \bar{J}^\mu_{\psi i}\, \psi_\mu{}^i + J^\mu_{U(1)_R}\, A_\mu^{U(1)_R} + \mathrm{tr}(J^\mu_{SU(2)_R}\, A_\mu^{SU(2)_R}) + J_T^{ab}\, T_{ab} + \bar{J}_{\chi i}\, \chi^i + J_D\, D \tag{9}$$

The currents $(J_e, J_\psi, J_{U(1)}, J_{SU(2)}, J_T, J_\chi, J_D)$ form the Sohnius multiplet.

[Sohnius 1979]

Supersymmetry is preserved iff the variation of the fermionic fields of the Weyl multiplet vanish

$$\delta_{\xi,\eta}\psi_\mu{}^i = 2D_\mu\xi^i + \frac{\mathrm{i}}{16}T_{ab}\gamma^{ab}\gamma_\mu\xi^i - \mathrm{i}\gamma_\mu\eta^i = 0 \tag{10}$$

$$\begin{aligned}\delta_{\xi,\eta}\chi^i = {} & \frac{\mathrm{i}}{24}\gamma^{ab}\gamma^\mu D_\mu T_{ab}\xi^i + \frac{1}{6}R(A^{SU(2)_R})_{ab}{}^i{}_j\gamma^{ab}\xi^j - \frac{1}{3}R(A^{U(1)})_{ab}\gamma^{ab}\gamma^5\xi^i + D\xi^i \\ & + \frac{1}{24}T_{ab}\gamma^{ab}\eta^i = 0\end{aligned} \tag{11}$$

$\hookrightarrow$ generalizes Killing spinor equation

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- X: a smooth, compact, oriented, 4-manifold with $b^+ \geq 1$
- $P \to X$: a principal G-bundle over X
- $M(P)$: the moduli space of irreducible anti-self-dual connections on P

Then the Donaldson invariant is a polynomial on $H_0(X) \oplus H_2(X)$ defined by

$$D_P(p^a, S^b) = \int_{M(P)} \mu(p)^a \wedge \mu(S)^b, \qquad p \in H_0(X),\ S \in H_2(X), \tag{12}$$

where $\mu : H_i(X) \to H^{4-i}(M(P))$ is the Donaldson map. We then formulate a generating function

$$Z_D(p, S) = \sum_{m,n \geq 0} \frac{D_P\left(p^n, S^m\right)}{n!m!}. \tag{13}$$

When $b_+ > 1$, Z_D is independent of the metric and defines invariants of the differentiable structure of X. When $b_+ = 1$, it is piecewise constant as a function of g. math/0311058

The twisted background

The topological twist may be implemented on the conformal supergravity by "turning on"

$$
\begin{aligned}
X &= S^3 \times S^1_\beta \\
A^{SU(2)_R} &= \frac{1}{2}\omega^{ab}\sigma_{ab} \\
D &= \frac{1}{6}R
\end{aligned}
\tag{14}
$$

This background preserves two constant Killing spinors

$$
(\xi_+)^{\alpha i} = (\sigma_2)^{\alpha i} \quad (\xi_-)^{\dot{\alpha} i} = (\sigma_2)^{\dot{\alpha} i} \tag{15}
$$

Now, $Z[g, A^{SU(2)_R}, D] = Z_{\text{twist}}[S^3 \times S^1_\beta]$.

Donaldson-Witten Twist

Consider a set of gauge-invariant operator densities in the twisted theory, $\mathcal{O}^{(n)}$, labelled by their degree. eg. Twisted SYM $\mathcal{O}^{(2)} = \frac{1}{2}\,\mathrm{Tr}[\phi F_{\mu\nu} + \psi_\mu \psi_\nu] dx^\mu \wedge dx^\nu$

We define the observable associated to a submanifold $\Sigma_n \subset X$ as

$$\mathcal{O}(\Sigma_n) = \int_{\Sigma_n} \mathcal{O}^{(n)}. \tag{16}$$

The following correlator is equal to the Donaldson polynomial

$$\langle (\mathcal{O}^{(0)}(p))^l (\mathcal{O}^{(2)}(S)),^r \rangle \tag{17}$$

and so are the correlation functions

$$Z_{\text{twist}}[p, S] = \langle e^{\mathcal{O}^{(0)}(p) + \mathcal{O}^{(2)}(S)} \rangle = Z_D(p, S). \tag{18}$$

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Trace definition

The Superconformal Index (SCI) is defined as the Witten index of the theory in radial quantization [Gadde, Rastelli, Razamat, Yan 2013]

$$\mathcal{I}(\mu_i) = \mathrm{Tr}_{\mathcal{H}(S^{d-1})} \left((-1)^F e^{-\beta\delta} \prod_{i=1}^{N} \mu_i^{\mathcal{M}_i} \right) \tag{19}$$

where $\delta = 2\left\{\mathcal{Q}, \mathcal{Q}^\dagger\right\}$.

Counts states annihilated by δ and is independent of β.

The 4d $\mathcal{N}=2$ superconformal algebra has 8 real (Q-)supercharges.

$$\mathcal{Q}_{1-} \quad \mathcal{Q}_{1+} \quad \mathcal{Q}_{2-} \quad \mathcal{Q}_{2+} \quad \tilde{\mathcal{Q}}_{1\dot{-}} \quad \tilde{\mathcal{Q}}_{1\dot{+}} \quad \tilde{\mathcal{Q}}_{2\dot{-}} \quad \tilde{\mathcal{Q}}_{2\dot{+}} \tag{20}$$

Choose, say, $\tilde{\mathcal{Q}}_{1\dot{-}}$, to compute the index.

The commutants of $\tilde{\delta}_{1\dot{-}}$ are: $\tilde{\delta}_{2\dot{+}}$, δ_{1+} and δ_{1-}.

$$\mathcal{I}(p,q,t) = \mathrm{Tr}_{\mathcal{H}(S^3)}\left((-1)^F p^{\frac{1}{2}\delta_{1+}} q^{\frac{1}{2}\delta_{1-}} t^{R+r} e^{-\beta\tilde{\delta}_{1\dot{-}}}\right) \tag{21}$$

Superconformal Index
$\mathcal{I}(p, q, t)$

$\tau \to 0$

$p = 0$

$q = t$

Coulomb-branch Index
$\mathcal{I}(\tau\sigma, \tau\rho, \tau^2)$
$\hookrightarrow$ dep. on ρ, σ only

Macdonald Index
$\mathcal{I}(0, q, t)$

$q = 0$

$q = t$

Hall-Littlewood Index
$\mathcal{I}(0, 0, t)$

Schur Index
$\mathcal{I}(p, q, q)$

Partition function definition

The SCI may also be defined as a partition function on $S^3 \times S^1_\beta$

$$Z_{S^3 \times S^1_\beta} = e^{-\beta E_{\text{Casimir}}} \mathcal{I} \tag{22}$$

The various deformations around the maximal supersymmetric configuration (i.e. round $S^3 \times S^1$) parameterize p, q and t.

$$\mathbb{R}^4 \quad \xrightarrow[\text{equiv.}]{\text{conf.}} \quad S^3 \times \mathbb{R} \quad \xrightarrow[\text{holo.}]{U(1)_R} \quad S^3 \times S^1_\beta$$

Can define many indices in that way $\hookrightarrow L(p,q) \times S^1$, half index, etc

The conformal supergravity background

Only "turn on" the following Weyl multiplet fields

$$\begin{aligned} g &= d\theta^2 + \sin^2(\theta)d\varphi^2 + \cos^2(\theta)d\tau^2 + \beta^2 dt^2 \\ A^{U(1)_R} &= -\beta dt \end{aligned} \tag{23}$$

In general, p, q and t are related to

$$\begin{pmatrix} \mathrm{Spin}(4) \\ SU(2) \end{pmatrix} \supset \begin{pmatrix} U(1)^2 \\ U(1) \end{pmatrix} \longleftrightarrow \begin{pmatrix} p, q \\ t \end{pmatrix}$$

The Coulomb branch limit

The fully refined SCI background preserves two Killing spinors.

Our background preserves four Killing spinors

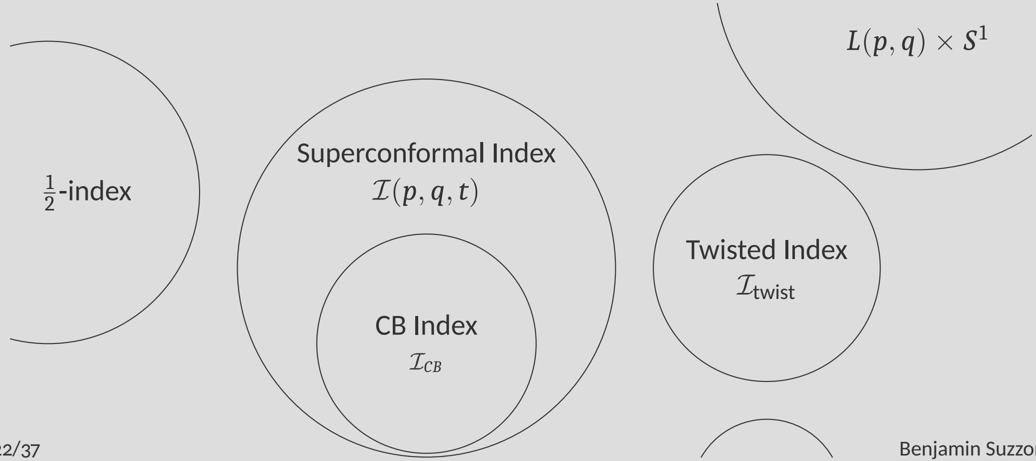
$$(\xi_+)^{\alpha i} = (\sigma_1)^{\alpha i} \quad (\xi_+)^{\alpha i} = (\sigma_2)^{\alpha i} \quad (\xi_-)^{\dot{\alpha} i} = (\sigma_1)^{\dot{\alpha} i} \quad (\xi_-)^{\dot{\alpha} i} = (\sigma_2)^{\dot{\alpha} i} \tag{24}$$

This engineers the the unrefined Coulomb branch index,

$$Z[g, A^{U(1)_R}] \propto \mathcal{I}_{CB} \tag{25}$$

The big picture

Zooming into the landscape of indices of 4d $\mathcal{N} = 2$ SCFTs



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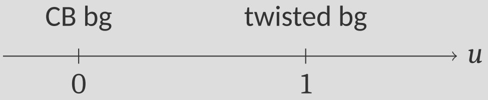
The conformal supergravity background

$$
\begin{aligned}
g &= d\theta^2 + \sin^2(\theta) d\varphi^2 + \cos^2(\theta) d\tau^2 + \beta^2 dt^2 \\
A^{U(1)_R} &= \beta(u-1)dt \\
A^{SU(2)_R} &= \frac{u}{2}\omega^{ab}\sigma_{ab} \\
D &= \frac{u(2-u)}{6}R
\end{aligned}
\tag{26}
$$

It interpolates between the Coulomb branch and the twisted backgrounds



This background preserves one Killing spinor, common to the CB and twisted backgrounds

$$(\xi_-)^{\dot{\alpha}i} = (\sigma_2)^{\dot{\alpha}i} \tag{27}$$

We will denote the corresponding partition function $\mathcal{I}_{\text{interp}}(u)$

$$\mathcal{I}_{\text{interp}}(u) = Z_{S^3 \times S^1_\beta}[A^{U(1)_R}, A^{SU(2)_R}, D] \tag{28}$$

$$\begin{tikzcd} & {} \arrow[dl, "u=0"'] \arrow[dr, "u=1"] & \\ \mathcal{I}_{CB} & & \mathcal{I}_{\text{twist}} \end{tikzcd}$$

We claim that the interpolation is *exact*

$$\frac{d\mathcal{I}}{du} = 0 \tag{29}$$

which in turn, means that we can equate the CB index and the twisted index.

$$\mathcal{I}_{CB} = \mathcal{I}_{\text{twist}} \tag{30}$$

*Ignoring all factors of the supersymmetric Casimir energy

Proof: Step 0

1. Pick bosonic fixed point of the supergravity bg: $\overline{\mathcal{H}}$
2. Expand around fixed point $\mathcal{H} = \overline{\mathcal{H}} + t\Delta\mathcal{H}$

$$S[\Phi; \overline{\mathcal{H}} + t\Delta\mathcal{H}] = S[\Phi; \overline{\mathcal{H}}] + t\sum_A \int (J_A, \Delta\mathcal{H}_A) + \mathcal{O}(t^2) \tag{31}$$

where $\mathcal{H}_A \in ({e_\mu}^a, {\psi_\mu}^i, b_\mu, A_\mu^{U(1)_R}, A_\mu^{SU(2)_R}, T_{ab}, \chi^i, D)$ and $J_A = \left.\frac{\delta S}{\delta \mathcal{H}_A}\right|_{t=0}$.

The curved space action is supersymmetric iff

$$\delta_{\xi,\eta} J_{\mathcal{H}_B} = -\sum_A \left(\frac{\partial \delta_{\xi,\eta}\mathcal{H}_A}{\partial \mathcal{H}_B} J_A - \partial_\mu \left(\frac{\partial \delta_{\xi,\eta}\mathcal{H}_A}{\partial \partial_\mu \mathcal{H}_B} J_A \right)\right) \tag{32}$$

Proof: Step 1

Show that dS/du is exact with respect to $\delta_{\xi,\eta}$

$$\frac{dS}{du} = \delta_{\xi,\eta} \int_{S^3 \times S^1_\beta} \left(J_{\chi i} \zeta^i + J_{\psi}{}_i^{\mu} \tilde{\zeta}^i_\mu \right), \tag{33}$$

where the spinors ζ^i and $\tilde{\zeta}^i_\mu$ must obey a set of equations in the supergravity fields. This stems from the fact that dS/du can be written as

$$\frac{d}{du} S[\mathcal{H}(u), \Phi] = \sum_A \int_{S^3 \times S^1_\beta} \frac{\delta S[\mathcal{H}(u), \Phi]}{\delta \mathcal{H}_A(u)} \frac{d\mathcal{H}_A(u)}{du}. \tag{34}$$

$$\frac{de^a{}_\mu}{du} = -\bar{\xi}_i\gamma^5\gamma^a\tilde{\zeta}^i_\mu \tag{35a}$$

$$\frac{dA^{U(1)}_\mu}{du} = \bar{\xi}_i\gamma_\mu\zeta^i + \frac{i}{2}\bar{\xi}_j\frac{\partial\phi^j_\mu}{\partial\psi^i_\nu}\tilde{\zeta}^i_\nu + \frac{i}{2}\bar{\eta}_i\tilde{\zeta}^i_\mu + \frac{1}{4}\bar{\xi}_i(\gamma^{\nu\rho}\gamma_\mu - \frac{1}{3}\gamma_\mu\gamma^{\nu\rho})\partial_\nu\tilde{\zeta}^i_\rho \tag{35b}$$

$$\vdots$$

Against all odds, those equations admit a solution

$$\zeta = \frac{i}{4}\begin{pmatrix} 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \tilde{\zeta}_\mu = c_\mu \begin{pmatrix} 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{36}$$

Proof: Step 2

Since $\frac{dS}{du} = \delta_{\xi,\eta}\mathcal{O}$ and $\delta_{\xi,\eta}S = 0$,

$$\begin{aligned}
\frac{d\mathcal{I}}{du} &= \frac{d}{du}\int \mathcal{D}\Phi e^{-S[\Phi,\mathcal{H};u]} \\
&= -\int \mathcal{D}\Phi\, \delta_\xi \mathcal{O}\, e^{-S[\Phi,\mathcal{H};u]} \\
&= -\int \mathcal{D}\Phi\, \delta_\xi \left(\mathcal{O}\, e^{-S[\Phi,\mathcal{H};u]}\right) \\
&= 0
\end{aligned}$$

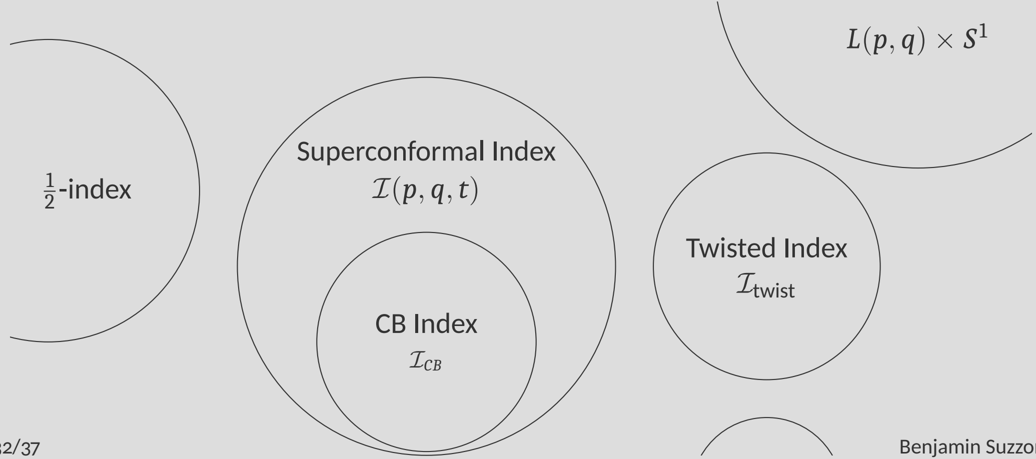
■

Example: vplet

$$
\begin{aligned}
e^{-1}\mathcal{L}_V = \mathrm{Tr}\Bigg(& D_\mu X_+ D^\mu X_- + \left(D + \frac{R}{6}\right) X_+ X_- - \frac{1}{4}\left(\bar{\Omega}_{-i}\gamma^\mu D_\mu \Omega^i_+ + \bar{\Omega}_{+i}\gamma^\mu D_\mu \Omega^i_-\right) \\
& + \sum_\pm \Bigg(\frac{1}{8} F^\pm_{ab} F^{\pm ab} \pm \frac{ig}{2} \bar{\Omega}_{\mp i}[X_\pm, \Omega^i_\mp] - \frac{1}{8} X_\pm F^\pm_{ab} T^{\pm ab} + \frac{1}{64} X^2_\pm T^\pm_{ab} T^{\pm ab} \\
& + \frac{1}{8} Y^{ij} Y_{ij} - g[X_-, X_+]^2 \Bigg)\Bigg) \qquad (37)
\end{aligned}
$$

Result

Zooming into the landscape of indices of 4d $\mathcal{N} = 2$ SCFTs



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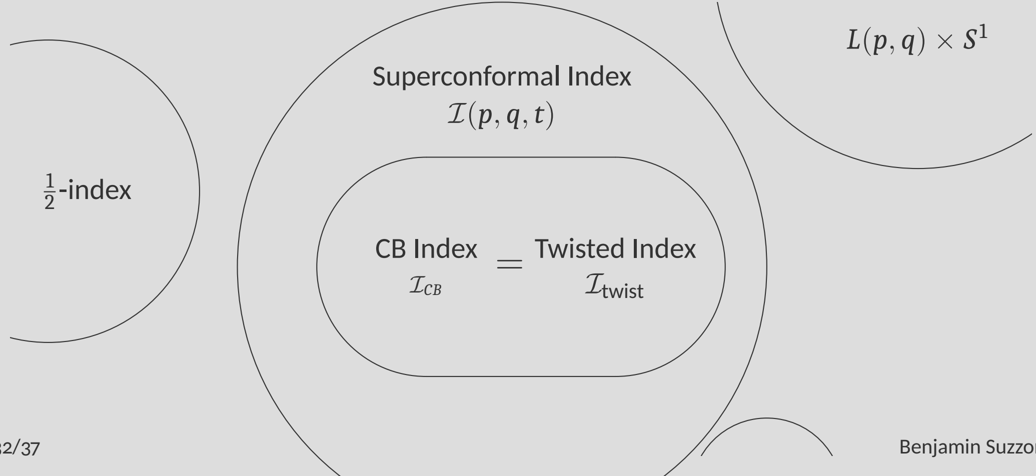

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Summary of Results

1. Constructed a supersymmetric background which interpolates between the Coulomb branch index and the twisted index
2. Demonstrated the exactness of the interpolation, allowing us to equate the indices
3. In fact, all partition functions at any value of u match
4. Constructed a general formula for the susy variation of the current multiplet in terms of that of the background sugra fields

Honorable mentions

1. Three-sphere squashing and primary Hopf-surfaces [Closset Dumitrescu Festuccia Komargodski 2014]
2. Interpolation between the refined CBI and the Ω-deformed twisted index?
   $\hookrightarrow$ which deformations preserve the interpolation?

Honorable mentions

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2. Interpolation between the refined CBI and the Ω-deformed twisted index?
   $\hookrightarrow$ which deformations preserve the interpolation?

$$\begin{tikzcd} {} \arrow[rr, leftrightarrow, "?"] & & {} \\ \mathrm{CB} \arrow[u, "{\rho, \sigma}"] \arrow[rr, leftrightarrow, "u"] & & \text{Twisted} \arrow[u, "\Omega\text{-deformation}"'] \end{tikzcd}$$

Implications and future directions

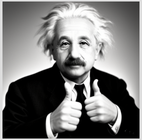
$$\mathcal{I}_{CB}(\rho,\sigma) \stackrel{?}{=} \mathcal{I}_{\text{twisted}}(\epsilon_1,\epsilon_2) \tag{38}$$

What does this result imply?

1. Holographically: BH microstate counting?
   - Kerr-Newman BH with angular momentum and electric charge
   - Rotating BH with magnetic R-charge
2. Holographically: exactness $\Rightarrow$ interpolating sugra solution??

An exact interpolating index for 4d $\mathcal{N} = 2$ SCFTs

Thank you for your attention



Benjamin Suzzoni

Local symmetry	Weyl multiplet field		$U(1)_r$ weight
Independent Connections			
Translations	$e^a{}_\mu$	$\in \Gamma(T\mathcal{M} \otimes T^*\mathcal{M})$	0
Dilatations	b_μ	$\in \Gamma(T^*\mathcal{M})$	0
$U(1)_r$ R-symmetry	$A_\mu^{U(1)_r}$	$\in \Gamma(T^*\mathcal{M})$	0
$SU(2)_R$ R-symmetry	$A_\mu^{SU(2)_R\, i}{}_k$	$\in \Gamma\left(T^*\mathcal{M} \otimes \mathrm{Sym}^2\, \mathcal{K}_R\right)$	0
Poincaré SUSY	ψ_μ^i	$\in \Gamma\left(T^*\mathcal{M} \otimes S\mathcal{M} \otimes \mathcal{K}_R\right)$	$\mp \frac{1}{2}$
Composite Connections			
Lorentz	$\omega_\mu{}^{ab}$	$\in \Gamma\left(T^*\mathcal{M} \otimes \Lambda^2 T\mathcal{M}\right)$	0
Conformal SUSY	ϕ_μ^i	$\in \Gamma\left(T^*\mathcal{M} \otimes S\mathcal{M} \otimes \mathcal{K}_R\right)$	$\pm \frac{1}{2}$
Special Conformal	$f_\mu{}^a$	$\in \Gamma\left(T^*\mathcal{M} \otimes T\mathcal{M}\right)$	0
Auxiliary Fields			
—	D	$\in \Omega^0(\mathcal{M})$	0
—	$T_{\mu\nu}$	$\in \Omega^2(\mathcal{M})$	± 1
—	χ^i	$\in \Gamma(S\mathcal{M} \otimes \mathcal{K}_R)$	$\mp \frac{1}{2}$

After twisting, we are still free to turn on supergravity fields at no further cost in susy,

$$\begin{aligned} ds^2 &= d\theta^2 + \sin^2(\theta)(d\varphi + \epsilon_1 dt)^2 + \cos^2(\theta)(d\tau + \epsilon_2 dt)^2 + \beta^2 dt^2 \\ A^{SU(2)_R} &= \frac{1}{2}\omega^{ab}\sigma_{ab} \quad T = 4(dv)^+ \quad D = \frac{R}{6} \end{aligned} \tag{39}$$

where $v = \epsilon_1 \partial_\varphi + \epsilon_2 \partial_\tau$.

The background now preserves

$$\begin{pmatrix} (\xi_+)^{\alpha i} \\ (\xi_-)^{\dot\alpha i} \end{pmatrix} = \begin{pmatrix} (\sigma_2)^{\alpha i} \\ \mathbb{i} v^\mu (\sigma_\mu \sigma_2)^{\dot\alpha i} \end{pmatrix} \quad (\xi_-)^{\dot\alpha i} = (\sigma_2)^{\dot\alpha i} \tag{40}$$

$\hookrightarrow U(1)^2$-equivariant twisted index

Relating susy generators after frame rotation

$$F_{ij} = \frac{1}{4}\mathrm{tr}(\gamma_i R \gamma_j R) \qquad F \in SO(4), R \in \mathsf{Spin}(4) \tag{41}$$

$$\xi^{(1)}_{\mathbb{R}^4} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \xi^{(2)}_{\mathbb{R}^4} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad \xi^{(3)}_{\mathbb{R}^4} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad \xi^{(4)}_{\mathbb{R}^4} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\xi^{(5)}_{\mathbb{R}^4} = -x_a \gamma^a \xi^{(1)} \quad \xi^{(6)}_{\mathbb{R}^4} = -x_a \gamma^a \xi^{(2)} \quad \xi^{(7)}_{\mathbb{R}^4} = -x_a \gamma^a \xi^{(3)} \quad \xi^{(8)}_{\mathbb{R}^4} = -x_a \gamma^a \xi^{(4)}$$

Relating susy generators after frame rotation

$$
\begin{aligned}
e^1 &= dx_1 = e^{-t\beta}\cos(\varphi)\sin(\theta)d\varphi + e^{-t\beta}\sin(\varphi)\cos(\theta)d\theta - \beta e^{-t\beta}\sin(\varphi)\sin(\theta)dt\\
e^2 &= dx_2 = -e^{-t\beta}\sin(\varphi)\sin(\theta)d\varphi + e^{-t\beta}\cos(\varphi)\cos(\theta)d\theta - \beta e^{-t\beta}\cos(\varphi)\sin(\theta)dt\\
e^3 &= dx_3 = -e^{-t\beta}\cos(\theta)\sin(\tau)d\tau - e^{-t\beta}\cos(\tau)\sin(\theta)d\theta - \beta e^{-t\beta}\cos(\theta)\cos(\tau)dt\\
e^4 &= dx_4 = -e^{-t\beta}\sin(\tau)\sin(\theta)d\tau + e^{-t\beta}\cos(\tau)\cos(\theta)d\theta - \beta e^{-t\beta}\sin(\tau)\sin(\theta)dt
\end{aligned}
$$

$$
R = \begin{pmatrix}
e^{-i\tau}\cos\theta & e^{-i\varphi}\sin\theta & 0 & 0\\
-e^{i\varphi}\sin\theta & e^{i\tau}\cos\theta & 0 & 0\\
0 & 0 & i & 0\\
0 & 0 & 0 & -i
\end{pmatrix} \in Spin(4)
$$

$$\frac{de^a{}_\mu}{du} = -\bar{\xi}_i\gamma^5\gamma^a\tilde{\zeta}^i_\mu \tag{42a}$$

$$\frac{db_\mu}{du} = \frac{1}{2}\bar{\xi}_i\gamma^5\gamma_\mu\zeta^i - \frac{i}{2}\bar{\xi}_i\gamma^5\frac{\partial\phi^i_\mu}{\partial\psi^j_\nu}\tilde{\zeta}^j_\nu - \frac{i}{2}\bar{\eta}_i\gamma^5\tilde{\zeta}^i_\mu - \frac{1}{4}\bar{\xi}_i\gamma^5(\gamma^{\nu\rho}\gamma_\mu - \frac{1}{3}\gamma_\mu\gamma^{\nu\rho})\partial_\nu\tilde{\zeta}^i_\rho \tag{42b}$$

$$\frac{dA^{U(1)}_\mu}{du} = \bar{\xi}_i\gamma_\mu\zeta^i + \frac{i}{2}\bar{\xi}_j\frac{\partial\phi^j_\mu}{\partial\psi^i_\nu}\tilde{\zeta}^i_\nu + \frac{i}{2}\bar{\eta}_i\tilde{\zeta}^i_\mu + \frac{1}{4}\bar{\xi}_i(\gamma^{\nu\rho}\gamma_\mu - \frac{1}{3}\gamma_\mu\gamma^{\nu\rho})\partial_\nu\tilde{\zeta}^i_\rho \tag{42c}$$

$$\begin{aligned}\frac{dA^{SU(2)}_\mu{}^i{}_j}{du} &= \bar{\xi}_l\gamma^5\gamma_\mu(2\delta^l{}_j\delta^i{}_k - \delta^l{}_k\delta^i{}_j)\zeta^k - 2i\bar{\xi}_j\gamma^5\frac{\partial\phi^i_\mu}{\partial\psi^k_\nu}\tilde{\zeta}^k_\nu + 2i\bar{\eta}_j\gamma^5\tilde{\zeta}^i_\mu \\ &\quad + i\delta^i{}_j(\bar{\xi}_k\gamma^5\frac{\partial\phi^k_\mu}{\partial\psi^l_\nu}\tilde{\zeta}^l_\nu - \bar{\eta}_k\gamma^5\tilde{\zeta}^k_\mu) - \bar{\xi}_j\gamma^5(\gamma^{\nu\rho}\gamma_\mu - \frac{1}{3}\gamma_\mu\gamma^{\nu\rho})\partial_\nu\tilde{\zeta}^i_\rho \\ &\quad + \frac{1}{2}\delta^i{}_j\bar{\xi}_k\gamma^5(\gamma^{\nu\rho}\gamma_\mu - \frac{1}{3}\gamma_\mu\gamma^{\nu\rho})\partial_\nu\tilde{\zeta}^k_\rho\end{aligned} \tag{42d}$$

$$\frac{dT_{ab}}{du} = -4i\bar{\xi}_i\gamma^5\gamma_{ab}\zeta^i + 8i\bar{\xi}_j\gamma^5\frac{\partial R(Q)_{ab}{}^j}{\partial\psi^i_\mu}\tilde{\zeta}^i_\mu + 8i\bar{\xi}_j\gamma^5\frac{\partial R(Q)_{ab}{}^j}{\partial\partial_\mu\psi^i_\nu}\partial_\mu\tilde{\zeta}^i_\nu \tag{42e}$$

$$\frac{dD}{du} = -\bar{\xi}_j\gamma^5\gamma^\mu\frac{\partial D_\mu\chi^j}{\partial\chi^i}\zeta^i - \bar{\xi}_i\gamma^5\gamma^\mu\partial_\mu\zeta^i \tag{42f}$$